C2 Exercise 7C (GP in real-world problems)

Note Title 27/04/2013

1 A population of ants is growing at a rate of 10% per year. If there were 200 ants in the initial population, write down the number after

a 1 year — 10% growth results in 110% after a year and 110% is the same as the decimal multiplier 1.10, so the number of ants after a year is $200 \times 1.1 = 220$

b 2 years — After another year $220 \times 1.1 = 242$

c 3 years — and a third $242 \times 1.1 = 266.2 \approx 266$

d 10 years — you could continue this way, but it's better to use

$$= 200 \times (1.1)^{10} = 518.74849202\ldots$$
$$\approx 519 \text{ ants}$$

(notice it's $200 \times (1.1)^{\underline{10}}$ not to the power 9. This is because u_1 was zero years and hence 10 years is, in fact, the eleventh term)

2 A motorcycle has four gears. The maximum speed in bottom gear is 40km h^{-1} and the maximum spee in top gear is 120km h^{-1}. Given that the maximum speeds in each successive gear form a geometric progression, calculate, in km h^{-1} to one decimal place, the maximum speeds in the two intermediate gears.

$$u_1 = 40 = a$$
$$u_2 = ? = ar$$
$$u_3 = ? = ar^2$$
$$u_4 = 120 = ar^3$$

use $\dfrac{u_4}{u_1} = \dfrac{ar^3}{a} \Rightarrow r^3 = \dfrac{120}{40} = 3$

$\Rightarrow r = \sqrt[3]{3}$

$r = 1.442249570\ldots$

$\Rightarrow$ second gear top speed $u_2 = 40 \times (3)^{1/3}$

$= 57.6899828\ldots$

$\simeq 58 \text{ km h}^{-1}$

and third gear top speed $u_3 = 40 \times (3)^{2/3}$

$= 83.203352922\ldots$

$\simeq 83 \text{ km h}^{-1}$

3 A car depreciates in value by 15% a year. If it is worth £11 054.25 after 3 years, what was its new price and when will it first be worth less than £5000?

tabulate the situation first

new	$u_1 = P$	where P is the new price
after 1yr	$u_2 = P \times 0.85$	since 15% depreciation leaves 85% left.
after 2yrs	$u_3 = P \times 0.85^2$	
after 3yrs	$u_4 = P \times 0.85^3 = 11054.25$	
...	...	
after n years	$u_{n+1} = P \times 0.85^n$	

Now tackle the first question

$$P \times 0.85^3 = 11054.25$$

$$\Rightarrow \quad P = \frac{11054.25}{0.85^3}$$

$$\Rightarrow \quad P = 18000$$

Its new price was £18000.

And then the second.

consider $\quad P \times 0.85^n = 5000 \quad$ after n years

then

$$0.85^n = \frac{5000}{18000} = \frac{5}{18}$$ where $P = 18000$ from above.

... now take logs to any base. Log base 10 is probably best since almost all scientific calculators have a button for that and we explored $\log_{10} x$ in C2 chapter 3.

$$\Rightarrow \quad \log_{10}\left(0.85^n\right) = \log_{10}\left(\frac{5}{18}\right)$$

You should remember the rule that says $\boxed{\log_a x^n \equiv n \log_a x}$ (C2 section 3.4) "Logs Power Law"

$$\Rightarrow \quad n \log_{10}(0.85) = \log_{10}\left(\frac{5}{18}\right)$$

$$\Rightarrow \quad n = \frac{\log_{10}\left(\frac{5}{18}\right)}{\log_{10}(0.85)} = 7.881751679$$

so the value will be <u>below</u> £5000 after 7.88 years.

4 The population decline in a school of whales can be modelled by a geometric progression. Initially there were 80 whales in the school. Four years later there were 40. Find out how many there will be at the end of the fifth year.

tabulate:

initial	u_1	$= 80$	$= a$
after 1 yr	u_2	$=$	$= ar$
after 2 yrs	u_3	$=$	$= ar^2$
3yrs	u_4	$=$	$= ar^3$
4yrs	u_5	$= 40$	$= ar^4$
5yrs	u_6	$= ?$	$= ar^5$ ← the 'unknown'

find 'r'

$$\frac{u_5}{u_1} = \frac{ar^4}{a} = r^4 \quad \text{but} \quad \frac{u_5}{u_1} = \frac{40}{80} = \frac{1}{2} \Rightarrow r = \sqrt[4]{1/2}$$

find u_6 for 5^{th} year

$$ar^5 = 80 \times \left(\frac{1}{2}\right)^{5/4}$$

$$= 33.63585661$$

$$\simeq 34 \text{ whales left.}$$

(think about the context: a decimal answer doesn't make sense)

5 Find which term in the progression 3, 12, 48, ... is the first to exceed 1 000 000.

tabulate

$$u_1 = a = 3$$

$$u_2 = ar = 12$$

$$\Rightarrow r = 4$$

$$u_n = ar^{n-1}$$

set up equation related to inequality:

we want $u_n > 1000000$

consider $ar^{n-1} = 1000000$

$$\Rightarrow \quad 3 \times 4^{n-1} = 1000\,000$$

$$\Rightarrow \quad 4^{n-1} = \frac{1\,000\,000}{3}$$

take logs (to any base you wish, but 10 or 'e' would be obvious)

$$\Rightarrow \quad (n-1)\log_{10}(4) = \log_{10}\left(\frac{1000\,000}{3}\right)$$

$$\Rightarrow \quad n-1 = \frac{\log_{10}\left(\frac{1000\,000}{3}\right)}{\log_{10}(4)}$$

$$\Rightarrow \quad n-1 = 9.173303034$$

$$\Rightarrow \quad n = 10.173303034$$

now revisit the question: does the context demand an integer?

$\Rightarrow$ the $\underline{\underline{11^{th}}}$ term is the first to exceed a million.

6 A virus is spreading such that the number of people infected increases by 4% per day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

day zero $\quad u_1 = 100$

day 1 $\quad u_2 = 100 \times 1.04$

day 2 $\quad u_3 = 100 \times 1.04^2$

day n $\quad u_{n+1} = 100 \times 1.04^n$

consider

$$100 \times 1.04^n = 1000$$

$$\Rightarrow \quad 1.04^n = 10$$

$$\Rightarrow \quad n \log_{10}(1.04) = 1$$

why do I know $\log_{10} 10 = 1$?

$$\Rightarrow \quad n = \frac{1}{\log_{10}(1.04)}$$

$$n = 58.70839431$$

so after 59 days there are in excess of 1000 infections.

7 I invest £A in the bank at an interest rate of 3.5% per annum. How long will it be before I double my money?

zero yrs $u_1 = A$

1 yr $u_2 = Ar = A(1.035)$

2 yrs $u_3 = Ar^2 = A(1.035)^2$

n yrs $u_{n+1} = Ar^n = A(1.035)^n$

consider

$u_{n+1} = 2A$ for some time n years afterwards

then $A(1.035)^n = 2A$

$$\Rightarrow \quad 1.035^n = 2$$

$$\Rightarrow \quad n \log_{10}(1.035) = \log_{10} 2$$

$$\Rightarrow \quad n = \frac{\log_{10} 2}{\log_{10}(1.035)}$$

* also works log base 'e' or log base 2 etc. Try it!

$$n = 20.14879168$$

Now think about context and rounding...

"It takes just over 20 years to double your money."

"It takes 20.15 years to double your money."

"It takes 20 years 55 days to double your money."

"After 21 years you've more than doubled your money."

Which answer do you prefer? Why?

8 The fish in a particular area of the North Sea are being reduced by 6% each year due to overfishing. How long would it be before the fish stocks are halved?

after zero years $u_1 = P$ (initial population is P)

after 1 year $u_2 = P \times 0.94$

after 2 yrs $u_3 = P \times (0.94)^2$

...

after n years $u_{n+1} = P \times (0.94)^n$

consider the situation in which fish stocks are halved:

$$u_{n+1} = \tfrac{1}{2}P$$

$$\Rightarrow \quad P \times (0.94)^n = \tfrac{1}{2}P$$

$$\Rightarrow \quad (0.94)^n = \tfrac{1}{2}$$

$$\Rightarrow \quad n \log_{10}(0.94) = \log_{10}(0.5)$$

$$\Rightarrow \quad n = \frac{\log_{10}(0.5)}{\log_{10}(0.94)}$$

$$\Rightarrow \quad n = 11.20230558$$

So the fish stocks will have halved after 11.2 years (11 years 74 days).

Explore more functional decay and growth problems in:

http://mathsurgery.wikispaces.com/Phys+A2+unit+5+section+9.5+-+radioactive+decay